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# Chiral fermions on the lattice

*Dedicated to the memory of L. O'Raifeartaigh*

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**Abstract**

We discuss topological obstructions to putting chiral fermions on an even dimensional lattice. The setting includes Ginsparg-Wilson fermions, but is more general. We prove a theorem which relates the total chirality to the difference of generalised winding numbers of chiral projection operators. For an odd number of Weyl fermions this implies that particles and anti-particles live in topologically different spaces.

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# 1 Introduction

The formulation of lattice theories with chiral fermions has been a long-standing problem [1],[2] which is closely related to the fermion doubling problem. It has been proven early on, that it is impossible to maintain chiral symmetry or define a chiral gauge theory on the lattice, if some basic assumptions are met. This is the celebrated Nielsen-Ninomiya (NN) no-go theorem [3]-[7]. It states that each left-handed fermion on a lattice must be accompanied by a right-handed fermion with the same quantum numbers. Thus, a lattice theory for a single Weyl fermion seems to be ruled out, but also a lattice realisation of chiral symmetry since the latter would assign opposite charges to left- and right-handed particles.

In turn, the Ginsparg-Wilson (GW) relation describes how close one can get to the naïve chiral symmetry in a lattice theory [8]. The derivation of the GW relation also highlights how the no-go theorem could be circumvented. The GW relation is derived from a block spin transformation of a lattice action which enjoys the naïve chiral symmetry. In this sense it defines the 'natural' quantum chiral symmetry on the lattice. For a long time, investigations of lattice theories with GW fermions have been hampered by the fact that no explicit example was known of a Dirac operator (in an interacting theory) obeying the GW relation. This gap was filled in [9] (and, less explicitly, in [10]). It also turned out that the action of GW fermions is invariant under a generalised chiral transformation of the fields [11],[12].

Let us now focus on how the no-go theorem is avoided: a chiral symmetry based on the GW relation fails to meet the key assumption of the NN theorem, namely that the symmetries act on the fermions $\psi$ and $\bar{\psi}$ as if they were (Dirac-) conjugates of each other. While this is the case in Minkowski space-time, $\psi$ and $\bar{\psi}$ have to be treated as independent fields in the Euclidean space-time used for the formulation of lattice theories. Consequently, $\psi$ and $\bar{\psi}$ can transform independently under symmetry transformations.

In the present contribution we study the properties of general chiral transformations on the lattice for the case of free fermions. We derive a statement about generalised chiral projections including those which are related to GW Dirac operators. To that end we consider Dirac operators $D$ which allow the definition of local chiral projections: $P_\psi$ and $P_{\bar{\psi}}$ with $DP_\psi = P_{\bar{\psi}}D$ and $P^2_{\psi/\bar{\psi}} = P_{\psi/\bar{\psi}}$. Apart from this, some technical conditions have to be met in order to ensure the vanishing of lattice artifacts in the continuum limit. Then the following statement can be proven:

$$\chi = n[P_\psi] - n[P_{\bar{\psi}}]$$

where $n[P_{\psi/\bar{\psi}}]$ is a (generalised) winding number of $P_{\psi/\bar{\psi}}$ defined in (7) and $\chi$ is the total chirality of all fermion species emerging in the continuum limit. It is also shown that the windings $n[P_{\psi/\bar{\psi}}]$ are integers. As a corollary this implies a version of the no-go theorem: for total chirality $\pm 1$ it is impossible to find chiral projections $P_{\psi/\bar{\psi}}$ with $P_{\bar{\psi}} = 1 - P_\psi$. The theorem proven here applies to even dimensional Euclidean lattices; in particular it covers the case of 4-dimensional Euclidean lattices instead of the 3-dimensional spatial lattices considered in [4, 5, 7]. Reference [6] deals with 4-dimensional lattices but restricts the form of the action rather strongly and does not allow for momentum dependent chiral projections. The latter, in particular, has important consequences. A first account of the present work was given in [13].

The paper is organised as follows. In the second section we state the theorem and some corollaries. We also discuss the necessity and implications of certain properties imposed on the Dirac operator and the projections. In the third section the winding numbers $n[P_{\psi/\bar{\psi}}]$ are evaluated and the theorem is proven. We close with a brief discussion of our findings. Some technical details are deferred to the appendices together with an example which highlights the difference between 3-dimensional (Hamiltonian) and 4-dimensional (Euclidean) lattices.

# 2 Theorem

We state the theorem in Sect. 2.1 and discuss its implications in Sect. 2.2.

## 2.1 Setting and theorem

We consider free, massless fermions on an infinite $d$-dimensional hyper-cubic lattice $\Lambda$ with lattice spacing $a$, $\Lambda = \{n_\mu a | n_\mu \in \mathbf{Z}\}$, where $d = 2l$ is even. At each lattice site, two $M$-component spinors $\psi(x), \bar{\psi}(x) \in \mathbf{C}^M$ are defined. The action is given in terms of a translationally invariant Dirac operator $D(x-y) \in \mathbf{C}^{M \times M}$,

$$S = \sum_{x,y \in \Lambda} \bar{\psi}(x) D(x-y) \psi(y) \; . \tag{1}$$

The Fourier transform of $D$,

$$D(k) \equiv \sum_{x \in \Lambda} \mathrm{e}^{-\mathrm{i}k \cdot x} D(x) \; , \tag{2}$$

is periodic with periods $2\pi/a$ on $\mathbf{R}^{2l}$ and can therefore be considered as a function from the $2l$-torus $T^{2l}$ to $\mathbf{C}^{M \times M}$. The spinors $\psi, \bar{\psi}$ live in the spaces defined by the constraints

$$P_\psi(k)\psi(k) = \psi(k) \qquad \text{and} \qquad \bar{\psi}(k) P_{\bar{\psi}}(k) = \bar{\psi}(k) \tag{3}$$

with translationally invariant Hermitean projection operators $P_{\psi/\bar{\psi}} = P^2_{\psi/\bar{\psi}}$. This includes the trivial case $P_{\bar{\psi}} = P_\psi = 1$. For a theory with chiral fermions, the projections $P_{\psi/\bar{\psi}}$ are necessarily non-trivial, as one has to remove part of the degrees of freedom. We would also like to emphasise that $P_\psi$ and $P_{\bar{\psi}}$ can be different since in a Euclidean theory $\psi$ and $\bar{\psi}$ are independent fields. We will prove the following:

**Theorem:** Given a lattice theory in $2l$ dimensions with the action (1) and projections $P_{\bar\psi}$, $P_\psi$ as in (3), where the Dirac operator $D$ and the projection operators $P_{\psi/\bar\psi}$ have the properties (i)-(iii):

(i) *locality:* $|D_{ij}(x)|, |P_{\bar\psi\, ij}(x)|, |P_{\psi\, ij}(x)| < c\,\mathrm{e}^{-|x|/\lambda}$ for some real constants $c, \lambda$. This implies that $D(k)$ and $P_{\psi/\bar\psi}(k)$ are analytic in $k_\mu$ in a strip around the real axis.

(ii) *spin-$\frac{1}{2}$ zeros:* the real poles $k^{(i)}$ of the propagator $D(k)^{-1}$ have the form

$$D(k)^{-1} = \frac{(k_\mu - k_\mu^{(i)})}{|k - k^{(i)}|^2}\,\Sigma_\mu^{(i)\dagger} + \text{finite}\,, \tag{4a}$$

$$\begin{aligned}\Sigma_\mu^{(i)\dagger}\Sigma_\nu^{(i)} + \Sigma_\nu^{(i)\dagger}\Sigma_\mu^{(i)} &= 2\delta_{\mu\nu}\,\Pi_\psi^{(i)} \quad \text{with} \quad \Pi_\psi^{(i)} = \Pi_\psi^{(i)2} = \Pi_\psi^{(i)\dagger}\,,\\ \Sigma_\mu^{(i)}\Sigma_\nu^{(i)\dagger} + \Sigma_\nu^{(i)}\Sigma_\mu^{(i)\dagger} &= 2\delta_{\mu\nu}\,\Pi_{\bar\psi}^{(i)} \quad \text{with} \quad \Pi_{\bar\psi}^{(i)} = \Pi_{\bar\psi}^{(i)2} = \Pi_{\bar\psi}^{(i)\dagger}\,,\end{aligned} \tag{4b}$$

(iii) *compatibility of chiral projections:* the chiral projection operators and the Dirac operator satisfy

$$D\,P_\psi = P_{\bar\psi}\,D\,. \tag{5}$$

Then the total chirality of all fermion species in the continuum limit is given by

$$\chi = n[P_\psi] - n[P_{\bar\psi}]\,. \tag{6}$$

where

$$n[P] \equiv \frac{1}{l!}\left(\frac{i}{2\pi}\right)^l \int_{T^{2l}} \mathrm{tr}\ P(\mathrm{d}P)^{2l} \in \mathbf{Z}\,. \tag{7}$$

Moreover $n[P]$ is a topological invariant with integer values for local, translationally invariant, Hermitean projection operators $P$ on the Fourier space $T^{2l}$. □

The theorem implies that

1. a non-zero total chirality is only possible with non-trivial projections ($P_\psi \neq 1$ or $P_{\bar\psi} \neq 1$ or both),
2. an odd total chirality is only possible if the spaces onto which $P_{\bar\psi}$ and $P_\psi$ project are neither identical nor orthogonal, i.e., $P_{\bar\psi} \neq P_\psi$ and $P_{\bar\psi} \neq 1 - P_\psi$, because otherwise $n[P_{\bar\psi}] = n[P_\psi]$ or $n[P_{\bar\psi}] = -n[P_\psi]$. The spaces onto which they project (and their orthogonal complements) are in fact inequivalent fibre bundles over $T^{2l}$.

The theorem does not exclude even non-zero chirality $\chi = 2n[P_\psi]$ if $P_{\bar\psi} = 1 - P_\psi$. This is in contrast to the situation for 3-dimensional spatial lattices, where only zero chirality is possible [4]-[7]. An example with $\chi = 4$ in 2 dimensions (and $\chi = 16$ in 4 dimensions) presented in App. A shows, that this is realised.

GW fermions [8] are included in (5) as they are defined with

$$\{\gamma_{2l+1},\, D\} = aD\,\gamma_{2l+1}\,D, \quad \text{where} \quad \gamma_{2l+1} = \mathrm{i}^l\gamma_1\cdots\gamma_{2l} \quad \text{with} \quad \gamma_{2l+1}^2 = 1. \tag{8}$$

Thus, they admit the (non-unique) definition of $P_{\psi/\bar\psi}$ with $P_\psi = \frac{1}{2}(1-\gamma_{2l+1})$ and $P_{\bar\psi} = \frac{1}{2}(1+\gamma_{2l+1}(1-aD))$. One can prove with (8) that $P_{\psi/\bar\psi}$ satisfy (5). Moreover $P_\psi^2 = P_\psi$ and $P_{\bar\psi}^2 = P_{\bar\psi}$. A similar analysis applies to Dirac operators satisfying a recently discussed generalisation of the GW relation [14].

## 2.2 Necessity and implications of the properties (i)-(iii)

Before proving the theorem in the next section, we first would like to elaborate a bit on the properties (i)-(iii):

Locality (i) of $D$ and the chiral projections guarantees that the continuum limit of the lattice theory does not depend on the details of the discretisation. In a local theory, only the behaviour at the zeros of $D$ matters in the continuum limit.

The structure of the zeros (ii) is determined by the requirement that the fields in the continuum limit carry spin-$\frac{1}{2}$ representations of the Euclidean group. A pole of $D^{-1}$ of the form (4) gives rise to a continuum action density $\bar\psi\, k\cdot\Sigma\,\psi$ (we suppress the superscript $(i)$ on $\Sigma$). $\Pi_\psi$ and $\Pi_{\bar\psi}$ project onto the right and left eigenspaces with vanishing eigenvalues of $D$, i.e. onto those components of $\psi$ and $\bar\psi$ that survive the continuum limit. Like the $\gamma$-matrices, the $\Sigma_\mu$ define spin-$\frac{1}{2}$ representations of the rotation group $SO(2l)$. The fermions $\bar\psi, \psi$ live in the two different representations generated by

$$\begin{aligned} \tfrac{1}{2}\Sigma^{\psi}_{\mu\nu} &\equiv \tfrac{1}{4}(\Sigma^\dagger_\mu\Sigma_\nu - \Sigma^\dagger_\nu\Sigma_\mu)\ , \\ \tfrac{1}{2}\Sigma^{\bar\psi}_{\mu\nu} &\equiv \tfrac{1}{4}(\Sigma_\mu\Sigma^\dagger_\nu - \Sigma_\nu\Sigma^\dagger_\mu)\ . \end{aligned} \tag{9}$$

The matrix $\Sigma_\mu$ couples the representations $\Sigma^{\bar\psi}$ and $\Sigma^\psi$ to a vector. Therefore, the continuum action $\bar\psi\, k\cdot\Sigma\,\psi$ is rotationally invariant. In turn, fermions in the spin representations (9) and the requirement of the correct continuum limit lead to a pole structure of $D^{-1}$ as in (4).

It follows from (4b) that we can write $\Sigma$ more explicitly as

$$\Sigma^{(i)}_\mu = U^{(i)}\ \mathrm{diag}[\underbrace{\sigma_\mu,\ldots,\sigma_\mu}_{n_+},\overbrace{\sigma^\dagger_\mu,\ldots,\sigma^\dagger_\mu}^{n_-}]\,V^{(i)\dagger}, \tag{10}$$

where the $\sigma_\mu$ and $\sigma^\dagger_\mu$ form right- and left-handed $2^{l-1}$-dimensional irreducible representations of (4b) without projections. These are unique up to bi-unitary transformations. Here, right-handed means $\mathrm{i}^l\sigma^\dagger_1\sigma_2\cdots\sigma^\dagger_{2l-1}\sigma_{2l} = +1$, which implies that

$\sigma^\dagger$ is left-handed, $\mathrm{i}^l \sigma_1 \sigma_2^\dagger \cdots \sigma_{2l-1} \sigma_{2l}^\dagger = -1$. For $2l = 4$, one can choose $\sigma_4 = 1$ and $\sigma_i = \mathrm{i}\tau_i$ where $\tau_i$ are the Pauli matrices. We also have

$$\Pi_{\bar\psi}^{(i)} = U^{(i)} U^{(i)\dagger}, \quad \Pi_{\psi}^{(i)} = V^{(i)} V^{(i)\dagger} \quad \text{with} \quad V^{(i)\dagger} V^{(i)} = U^{(i)\dagger} U^{(i)} = 1_{2^{l-1}(n_+ + n_-)} . \tag{11}$$

Thus, a pole of the form (4) gives rise to $n_+$ right- and $n_-$ left-handed fermions in the continuum limit, where $n_+$ and $n_-$ are the number of $\sigma_\mu$ and $\sigma_\mu^\dagger$ in $\Sigma_\mu^{(i)}$ respectively. The corresponding components of $\psi$ and $\bar\psi$ respectively are obtained as eigenspaces for eigenvalues $\pm 1$ of the chirality operators

$$\Gamma_\psi^{(i)} \equiv \mathrm{i}^l\, \Sigma_1^{(i)\dagger} \Sigma_2^{(i)} \cdots \Sigma_{2l-1}^{(i)\dagger} \Sigma_{2l}^{(i)} , \qquad \Gamma_{\bar\psi}^{(i)} \equiv \mathrm{i}^l\, \Sigma_1^{(i)} \Sigma_2^{(i)\dagger} \cdots \Sigma_{2l-1}^{(i)} \Sigma_{2l}^{(i)\dagger} . \tag{12}$$

The $\Gamma_{\psi/\bar\psi}^{(i)}$ are Hermitean and have eigenvalues $\pm 1$ and 0. Taking into account the projection (3), the total chirality of all fermion species in the continuum limit is given by

$$\chi = \frac{1}{2^{l-1}} \sum_i \mathrm{tr}\big[P_\psi(k^{(i)})\, \Gamma_\psi^{(i)}\big] . \tag{13}$$

The possibility of vector-like (Dirac) zeros is contained in (ii): if the $\Sigma_\mu$ are Hermitean, (4b) turns into the standard anti-commutation relations for $\gamma$-matrices in the image of $\Pi_\psi = \Pi_{\bar\psi}$.

The compatibility condition (iii) for $D$ and the chiral projections is, for local $P_{\psi/\bar\psi}$, equivalent to the following compatibility condition for $P_\psi$ and the spin representations $\Sigma^{\psi\,(i)}$ at the zeros $k^{(i)}$,

$$\big[\Sigma_{\mu\nu}^{\psi\,(i)}, P_\psi(k^{(i)})\big] = 0 \,; \tag{14}$$

the form of $P_{\bar\psi}$ then follows from that of $P_\psi$ via (5). Hence the property (5) is, at its root, only a constraint at the points $k^{(i)}$. It is for this reason, that the proof of the theorem boils down to the calculation of windings at these points. The relation (14) ensures that the projected spinor $P_\psi \psi$ also transforms under $SO(4)$; it contains complete irreducible components of the representation $\Sigma^{\psi\,(i)}$ only. It also follows that

$$\big[\Pi_\psi,\, P_\psi(k^{(i)})\big] = 0. \tag{15}$$

To prove that (5) implies (14) and (15), first note that it implies $P_\psi D^{-1} = D^{-1} P_{\bar\psi}$. Since $P_{\psi/\bar\psi}$ are analytic, we can expand in powers of $q \equiv k - k^{(i)}$ to get

$$P_\psi(k^{(i)})\, \Sigma_\nu^\dagger = \Sigma_\nu^\dagger\, P_{\bar\psi}(k^{(i)}) \tag{16}$$

for all $\nu$, where we have used that we are free to choose $q_\mu = \delta_{\mu\nu}|q|$ (we have suppressed the superscript $(i)$ on $\Sigma$ again). Using (16) and its Hermitean conjugate one concludes that $\Sigma_\mu^\dagger\Sigma_\nu P_\psi(k^{(i)}) = P_\psi(k^{(i)})\Sigma_\mu^\dagger\Sigma_\nu$. With (4b) and (9) this leads to (14) and (15).

Conversely, for any local $P_\psi$ satisfying Eq. (14), we can define $P_{\bar\psi} \equiv DP_\psi D^{-1}$. Except for the zeros of $D$, analyticity of $P_{\bar\psi}$ follows in the set where $P_\psi$ and $D$ have this property. Equations (14) and (4b) imply $P_\psi(k^{(i)})\Sigma_\mu^\dagger = P_\psi(k^{(i)})\Sigma_\mu^\dagger\Sigma_\nu\Sigma_\nu^\dagger = \Sigma_\mu^\dagger\Sigma_\nu P_\psi(k^{(i)})\Sigma_\nu^\dagger$ for $\nu \neq \mu$ (no sum). As $D\Sigma_\mu = \mathcal{O}(k)$, the poles of $D^{-1}$ drop out and $P_{\bar\psi}$ is finite and analytic there as well. It goes without saying that similar statements and relations like (14), (15), (16) follow for $\Sigma_{\mu\nu}^{\bar\psi}$, $\Pi_{\bar\psi}$, $P_{\bar\psi}(k^{(i)})$.

Local projection operators $P_{\psi/\bar\psi}$ satisfying (5) are also relevant in non-chiral theories where the constraints (3) are not needed: they can be used to define charges $Q_\psi \equiv 1 - 2P_\psi$, $Q_{\bar\psi} \equiv 2P_{\bar\psi} - 1$ and a 'chiral' symmetry

$$\begin{aligned} \psi &\to \mathrm{e}^{\mathrm{i}\alpha Q_\psi}\psi \, , \\ \bar\psi &\to \bar\psi\, \mathrm{e}^{\mathrm{i}\alpha Q_{\bar\psi}} \, . \end{aligned} \tag{17}$$

The existence of such a symmetry with local charges, however, implies the existence of local projections only if their eigenvalues are non-zero and non-degenerate in the entire Brillouin zone, for instance if the charges are integer-valued. The theorem presented in this paper applies to such a symmetry as well. It implies that a symmetry of this kind that goes over to the standard chiral symmetry in the continuum limit necessarily acts on $\psi$ and $\bar\psi$ in an asymmetric way (not as if $\psi$ and $\bar\psi$ were Dirac conjugates of each other). Note, however, that symmetries with charges whose eigenvalues vanish somewhere can be useful in non-chiral theories, e.g. [11], [14]. The theorem does not make a statement about these symmetries. Indeed, the symmetries used in [11, 14], essentially, lead to projections that satisfy $P_{\bar\psi} = 1 - P_\psi$ as well as (5) although the image of $P_\psi$ contains only a single left-handed fermion. These projections are discontinuous at some points in momentum space, so they are not local.

The chirality of a fermion in the continuum limit is determined by $\Gamma_{\psi/\bar\psi}^{(i)}$ in (12) rather than $P_\psi$ or $P_{\bar\psi}$. The images of the latter may well contain fermions of different chirality. Hence, our setting includes theories with any number of left- and right-handed fermions (in particular vector-like theories for which we can set $P_\psi = P_{\bar\psi} = 1$). If only one chirality is desired, one has to require that $\Gamma_{\psi/\bar\psi}^{(i)}$ has a definite sign in the image of $P_\psi(k^{(i)})$. This is not necessary for our theorem, so we do not make this requirement. However, we still use the term 'chiral' projections for $P_{\psi/\bar\psi}$ since they treat left- and right-handed fermions differently in general.

Accordingly, also the charges $Q_{\psi/\bar\psi}$ need not coincide with the chirality operator $\Gamma_{\psi/\bar\psi}^{(i)}$ at the zeros of $D$. Equation (14) only guarantees that $Q_\psi(k^{(i)})$ and $\Gamma_\psi^{(i)}$ are simultaneously diagonalisable, as are $Q_{\bar\psi}(k^{(i)})$ and $\Gamma_{\bar\psi}^{(i)}$. Equation (17) goes over to the standard chiral symmetry in the continuum limit only if $\Gamma_{\psi/\bar\psi}^{(i)}$ has only the eigenvalues 1 and 0 in the image of $P_{\psi/\bar\psi}(k^{(i)})$.

We close the section with an explicit – and relevant – example, the overlap Dirac operator [9] in four dimensions. It is a GW Dirac operator, see (8), and, for vanishing gauge field, it is given by

$$aD = 1 - \cos\theta + \mathrm{i}\gamma \cdot \hat{p} \sin\theta \tag{18}$$

where $\hat{p} = p/|p|$, $p_\mu(k)$ and $\theta(k)$ are periodic functions in momentum space and $\gamma_\mu$ a (fixed) set of Dirac matrices. Local chiral projections can be defined as $P_{\bar{\psi}} = \frac{1}{2}(1+\gamma_5)$ and $P_\psi = \frac{1}{2}(1 - Q_\psi)$ where [12]

$$Q_\psi = \gamma_5(1 - aD) = \gamma_5 \cos\theta - \mathrm{i}\gamma_5\gamma \cdot \hat{p} \sin\theta \,. \tag{19}$$

Regarding $-\mathrm{i}\gamma_5\gamma_\mu$ and $\gamma_5$ as basis vectors in a 5-dimensional space, $Q_\psi$ takes values on the unit sphere in this space. The form of Eq. (7) suggests that $n[P_\psi]$ measures the degree, or winding number, of the map $Q_\psi\colon T^4 \to S^4$. The degree can be expressed as the number of times a fixed point on the target is taken, weighted by the orientation (the sign of the Jacobian), provided the Jacobian does not vanish at the chosen point. We may choose the point $Q_\psi = \gamma_5$, i.e., $\theta = 0$. This corresponds to the zeros of $D$. If $D$ has only a single zero, $Q_\psi$ has unit winding number. For the overlap operator this can be explicitly verified by studying the functions $\theta$ and $p_\mu$. So we find $n[P_\psi] = -1$ and $n[P_{\bar{\psi}}] = 0$ and verify the theorem (6). For general functions $\theta(k)$ and $p_\mu(k)$, the orientation of a zero is given by (minus) the winding number of $\hat{p}\colon S^3 \to S^3$ around the zero. Since this coincides with the definition of chirality in (13), the theorem holds true.

# 3 Proof of the theorem

First, in Sect. 3.1, we prove that the winding number $n[P]$ is an integer. Then, in Sect. 3, we show that $n[P_\psi] - n[P_{\bar{\psi}}]$ in (6) can be transformed into the formula for the total chirality as given on the rhs of (13).

## 3.1 The winding number $n[P]$

The power of the theorem depends crucially on the fact that $n[P]$ is an integer for local projection operators $P(k)$. Hence, before tackling the proof of the relation (6) we argue that $n[P] \in \mathbf{Z}$. This discussion will also shed some light on the interpretation of the invariant $n[P]$. To that end we express $P$ in terms of an orthonormal basis $\Psi = (\psi_1, ..., \psi_N) \in \mathbf{C}^{M\times N}$ (where $N = \operatorname{tr} P$ is the rank of $P$),

$$P \equiv \Psi\Psi^\dagger \qquad \text{with} \qquad \Psi^\dagger\Psi = 1_N \,. \tag{20}$$

Then, a general basis is given by $\Psi^v \equiv \Psi\, v$ with $v \in \mathrm{U}(N)$. Since $P$ is periodic in $k_\mu$, $\Psi$ in general satisfies the boundary conditions

$$\Psi(k + \hat{a}_\mu) = \Psi(k)\, u_\mu(k) \qquad \text{with} \qquad u_\mu(k) \in \mathrm{U}(N) \quad \text{and} \quad (\hat{a}_\mu)_\nu = \tfrac{2\pi}{a}\delta_{\mu\nu} \,. \tag{21}$$

The transition functions $u_\mu$ defined in this way satisfy the cocycle conditions

$$u_\mu(k)\, u_\nu(k+\hat{a}_\mu) = u_\nu(k)\, u_\mu(k+\hat{a}_\nu). \tag{22}$$

They define a U($N$) fibre bundle over $T^{2l}$. If this fibre bundle is non-trivial, the eigenspace of $P$ does not admit a globally smooth basis. We will see that $n[P]$ measures (part of) this non-triviality.

To characterise the fibre bundle, we define the U($N$) gauge potential and field strength

$$A = \Psi^\dagger \, \mathrm{d}\Psi \;, \qquad F = \mathrm{d}A + A \wedge A \;. \tag{23}$$

They obey the expected boundary conditions

$$A(k+\hat{a}_\mu) = u_\mu^\dagger(k)\, \big(A(k) + \mathrm{d}\big)\, u_\mu(k) \;, \tag{24}$$

$$F(k+\hat{a}_\mu) = u_\mu^\dagger(k)\, F(k)\, u_\mu(k) \;. \tag{25}$$

It follows from (20) and (23) that, for even dimension $d = 2l$, the winding number $n[P]$ as defined in (7) is given by the integral of the $l$th Chern character (cf. App. C):

$$n[P] = \int \mathrm{ch}_l(F) \;. \tag{26}$$

In general, this is not an integer but only a multiple of $1/l!$. For U($N$) bundles over $T^{2l}$, however, it is an integer. This follows directly from the Atiyah–Singer index theorem. There, the $l$th Chern character for a torus U($N$)-bundle is shown to be identical to the index of the related Dirac operator, which is an integer.

In the light of the discussion above there is a natural interpretation of (5) as a map between inequivalent U($N$)-bundles over the torus $T^{2l}$. For highlighting this fact and for later use in Sect. 3.2 let us discuss this in more detail. On $T^{2l}_{\mathrm{r}} \equiv T^{2l} \setminus \{k^{(i)}\}$ we can write Eq. (5) as

$$P_{\bar\psi} = \varepsilon(D)\, P_\psi\, \varepsilon^\dagger(D) \quad \text{with} \quad \varepsilon(D) \equiv D(D^\dagger D)^{-1/2} \;. \tag{27}$$

$\varepsilon(D)$ is unitary. In (27) we have used that $[P_\psi, (D^\dagger D)^{-1/2}] = 0$, which follows directly from (5) and its Hermitean conjugate. Two orthonormal bases $\Psi_{\bar\psi}$ of $P_{\bar\psi}$ and $\Psi_\psi$ of $P_\psi$ are therefore related by

$$\Psi_{\bar\psi} = \varepsilon(D) \Psi_\psi g \quad \text{with} \quad g = \Psi_\psi^\dagger \varepsilon^\dagger(D) \Psi_{\bar\psi} \in \mathrm{U}(N) \;. \tag{28}$$

The function $g$ is continuous except at the zeros $k^{(i)}$ of $D$ where $\varepsilon(D)$ is ill-defined. It is not periodic but satisfies the boundary conditions

$$g(k+\hat{a}_\mu) = u_\mu^{\psi\,\dagger}(k)\, g(k)\, u_\mu^{\bar\psi}(k) \;. \tag{29}$$

where $u_\mu^\psi$ and $u_\mu^{\bar\psi}$ are the transition functions of $\Psi_\psi$ and $\Psi_{\bar\psi}$. These are thus related by

$$u_\mu^{\bar\psi}(k) = g^\dagger(k)\, u_\mu^\psi(k)\, g(k+\hat a_\mu)\ . \tag{30}$$

Hence, if $g$ (restricted to the boundary) carries non-trivial topology, the two sets of transition functions $u^{\psi/\bar\psi}$ define inequivalent $\mathrm{U}(N)$-bundles. However, since $g$ is smooth outside of the zeros of $D$, its non-trivial content can be extracted from its windings at $k^{(i)}$, which are introduced by $\varepsilon(D)$.

## 3.2 Proof of equation (6)

The discussion in the previous section already suggests that the difference $n[P_{\bar\psi}] - n[P_\psi]$ is directly related to a homotopy class of the map $\varepsilon(D)$ around the zeros of $D$. Indeed, we shall see that it is given by

$$n[P_{\bar\psi}] - n[P_\psi] = \sum_i \nu_i[g] \tag{31}$$

where

$$\nu_i[g] = \lim_{\epsilon\to 0} b_{2l-1} \int_{\|k-k^{(i)}\|=\epsilon} \mathrm{tr}\,(g^{-1}\mathrm{d}g)^{2l-1} \quad \text{with } b_{2l-1} = (-1)^{l-1}\frac{(l-1)!}{(2l-1)!}\left(\frac{i}{2\pi}\right)^l \tag{32}$$

is the $\pi_{2l-1}$ winding of $g \in \mathrm{GL}(N)$ about $k^{(i)}$. The gauge function $g$ is given in terms of $\varepsilon(D)$ by (28).

From (31) only a few technical steps have to be invoked in order to prove (6). For the proof of (31) we resort to the fact that the difference of winding numbers is given by the integral of the difference of Chern character $\mathrm{ch}_l$, see (26):

$$n[P_{\bar\psi}] - n[P_\psi] = \int \mathrm{ch}_l[F_L] - \int \mathrm{ch}_l[F_R], \tag{33}$$

where $F_L, F_R$ are the field strengths of $A_{\bar\psi} = A[P_{\bar\psi}]$ and $A_\psi = A[P_\psi]$ respectively. The integral over a Chern character is a topological invariant of the underlying fibre bundle. It can therefore be calculated with any gauge field obeying the same boundary conditions, in particular we can replace $A_{\bar\psi}$ by $\tilde A_{\bar\psi} = A_\psi^g \equiv g^{-1}(A+\mathrm{d})g$. Now we use that $\mathrm{ch}_l$ can (locally) be related to its Chern-Simons form $\mathrm{cs}_{2l-1}$ by $\mathrm{ch}_l[F] = \mathrm{d}\,\mathrm{cs}_{2l-1}[A,F]$. Furthermore

$$\mathrm{cs}_{2l-1}[A^g, F^g] - \mathrm{cs}_{2l-1}[A,F] = \mathrm{cs}_{2l-1}[g^{-1}\mathrm{d}g, 0] + \mathrm{d}\alpha_{2l-2}[A, F, \mathrm{d}g\, g^{-1}]\ . \tag{34}$$

Upon differentiation, the second term drops out. With $\mathrm{cs}_{2l-1}[A,0] = b_{2l-1}\,\mathrm{tr}\,A^{2l-1}$ (see e.g. [15], page 400) and integrating over the Brillouin zone $I^{2l} \equiv [-\pi/a, \pi/a]^4$, we find

$$n[P_{\bar\psi}] - n[P_\psi] = \nu[g], \tag{35}$$

where

$$\nu[g] = b_{2l-1} \int_{\partial I^{2l}} \mathrm{tr}\,(g^{-1}\mathrm{d}g)^{2l-1} \tag{36}$$

is the total $\pi_{2l-1}$ winding on $\partial I^{2l}$. Thus, (31) follows with the remark, that $g$ is continuous except for the zeros of $D$ and the density of $\nu$ is closed. Then, $\nu[g]$ equals the sum of the winding numbers of $g$: $\nu[g] = \sum_i \nu_i[g]$.

It is left to relate $\nu_i[g]$ to the chirality $\chi$. First note that in $g = \Psi_\psi^\dagger \varepsilon^\dagger(D)\Psi_{\bar\psi}$ one can replace $\Psi_\psi$ and $\Psi_{\bar\psi}$ by their values at $k^{(i)}$ and $\varepsilon^\dagger(D)$ by $D^{-1}$,

$$\nu_i[g] = \nu_i[g_i] \quad \text{with} \quad g_i = \Psi_\psi^\dagger(k^{(i)}) D^{-1} \Psi_{\bar\psi}(k^{(i)}) \,. \tag{37}$$

The first replacement is allowed because the bases $\Psi_{\psi/\bar\psi}$ can be chosen with finite derivatives (at least locally) and higher-order terms drop out in Eq. (32) in the limit $\varepsilon \longrightarrow 0$; the second because the set of positive definite Hermitean matrices is contractible, so $\sqrt{D^\dagger D}$ can be deformed to 1.

Now we employ the explicit form of $D$ about its zeros $k^{(i)}$. Using (5) and the cyclicity of the trace we arrive at

$$\mathrm{tr}\,(g_i^{-1}\,\mathrm{d}g_i)^{2l-1} = -\,\mathrm{tr}\,\left(P_\psi(k^{(i)})J\right)^{2l-1}, \quad \text{where} \quad J \equiv D^{-1}\,\mathrm{d}D. \tag{38}$$

Intuitively one expects that only those parts of $D$ and $D^{-1}$ can contribute to $\nu_i[g_i]$ that carry the Dirac structure $q \cdot \Sigma^{(i)}$ and $|q|^{-2}\, q \cdot \Sigma^{(i)\dagger}$, respectively ($q \equiv k - k^{(i)}$). Indeed we find

$$\nu_i[g_i] = -\lim_{\varepsilon\to 0} b_{2l-1} \int_{\|k-k^{(i)}\|=\varepsilon} \mathrm{tr}\;P_\psi(k^{(i)}) {J_0}^{2l-1}, \tag{39}$$

where

$$J_0 \equiv \frac{1}{|q|^2}\, q \cdot \Sigma^{(i)\dagger}\, \mathrm{d}q \cdot \Sigma^{(i)} \,. \tag{40}$$

As the derivation of (39) is a bit technical we defer it to Appendix B. For performing the integration in (39) we use the symmetry properties of the integral. We infer from (40) that

$$J_0^{2l-1} = \frac{1}{|q|^{2l}} \left( (-1)^{l-1} q \cdot \Sigma^{(i)\dagger}\, \mathrm{d}q \cdot \Sigma^{(i)} \left(\mathrm{d}q \cdot \Sigma^{(i)\dagger}\, \mathrm{d}q \cdot \Sigma^{(i)}\right)^{l-1} + \mathcal{O}(q\cdot \mathrm{d}q) \right) \tag{41}$$

where the algebra (4b) has been used and $\mathcal{O}(q\cdot\mathrm{d}q)$ denotes terms containing $q\cdot\mathrm{d}q$ as a factor. These do not contribute when integrated over 3-spheres centred at $q=0$. The first term yields

$$\lim_{\varepsilon\to 0} b_{2l-1} \int_{|k-k^{(i)}|=\varepsilon} J_0^{2l-1} = \frac{\mathrm{i}^l}{2^{l-1}} \frac{1}{(2l)!} \epsilon_{\mu_1\cdots\mu_{2l}}\, \Sigma_{\mu_1}^{(i)\dagger} \Sigma_{\mu_2}^{(i)} \cdots \Sigma_{\mu_{2l-1}}^{(i)\dagger} \Sigma_{\mu_{2l}}^{(i)} = \frac{1}{2^{l-1}} \Gamma_\psi^{(i)} \,. \tag{42}$$

By inserting (42) into (39) and the latter into (37), we obtain

$$n[P_{\bar\psi}] - n[P_\psi] = -\frac{1}{2^{l-1}} \sum_i \mathrm{tr}\left[P_\psi(k^{(i)})\Gamma_\psi^{(i)}\right] = -\chi \,. \tag{43}$$

Thus $n[P_\psi] - n[P_{\bar\psi}]$ is given by the total chirality of all fermion species appearing in the eigenspace of $P_\psi$. ■

For the proof we have employed some cohomology theory, many readers might be unfamiliar with. It is possible to avoid the use of (34) at the expense of tedious calculations. The integrand of the difference $n[P_{\bar\psi}] - n[P_\psi]$ is a total derivative $d\omega_{2l-1}$, which can be explicitly calculated by using $P_{\bar\psi} = D^{-1} P_\psi D$ on $T_{\mathrm{r}}^{2l} = T^{2l} \setminus \{k^{(i)}\}$, following from (5). Then $n[P_{\bar\psi}] - n[P_\psi] = \int_{\partial T_{\mathrm{r}}^{2l}} \omega_{2l-1}$. In four dimensions ($l = 2$) it follows after a straightforward, but rather lengthy calculation, that on $T_{\mathrm{r}}^{2l}$, the integrand in $n[P_{\bar\psi}] - n[P_\psi]$ can be written as

$$\mathrm{d}\omega_3 = \mathrm{tr}\, D\, P_\psi\, D^{-1}\big(\mathrm{d}(D\, P_\psi\, D^{-1})\big)^4 - \mathrm{tr}\, P_\psi (\mathrm{d}P_\psi)^4 \tag{44}$$

with

$$\omega_3 \equiv -\,\mathrm{tr}\left[J\,(J\,P_\psi)^2 - \tfrac{2}{3}(J\,P_\psi)^3 - 2(J\,P_\psi)^2 \mathrm{d}P_\psi + J\,P_\psi\,J\,\mathrm{d}P_\psi - 2J\,P_\psi(\mathrm{d}P_\psi)^2\right]. \tag{45}$$

Then, one proceeds by exploiting the properties of $J$ on $\partial T_{\mathrm{r}}^{2l} = \bigcup_i \{k|\, \|k - k^{(i)}\| = \epsilon\}$ as discussed in Appendix B. Since $P_\psi$ is smooth and $J$ has only linear singularities, only the first two terms on the rhs of (45) can contribute. On $\partial T_{\mathrm{r}}^{2l}$ it follows that $\omega_3|_{\partial T_{\mathrm{r}}^{2l}} = -\frac{1}{3}\,\mathrm{tr}\, P_\psi(k) J_0^3 + \mathcal{O}(q^{-2})$. Then one proceeds as from (39). We add that even deriving (45) is quite tedious and it gets increasingly complicated when going to higher dimensions. Moreover, within this approach the underlying topological structure gets obscured.

# 4 Conclusions

We have investigated the topological obstructions to implementing chiral symmetry on Euclidean lattices in general even dimensions. Our findings are summarised in section 2 in terms of a theorem. Its setting allows for general *local* chiral projections and Dirac operators, which, as a specific case, includes Ginsparg-Wilson fermions [8]. Within this setting the total chirality $\chi$ is given by the difference of winding number $n[P_\psi]$ and $n[P_{\bar\psi}]$:

$$\chi = n[P_\psi] - n[P_{\bar\psi}] \,.$$

Here, $P_{\psi/\bar\psi}$ are the projection operators defining the spaces of fermions: $P_\psi\psi = \psi$, $\bar\psi P_{\bar\psi} = \bar\psi$, see (3). The invariants $n[P]$ were shown to be integers.

This constitutes a generalisation of the Nielsen-Ninomiya no-go theorem. Let us briefly recapitulate the setting and the consequences of the original theorem and its variants. In [3, 4, 5, 7] Hamiltonian (spatial) lattices and general symmetric projections are considered, whereas [6] deals with Euclidean lattices and constant projections. Then, in both settings, the no-go theorem states that a non-vanishing total chirality $\chi$ is excluded.

In turn, the present formulation of the theorem implies that the projections $P_{\bar{\psi}}$ and $1-P_{\psi}$ have to be topologically inequivalent for odd chirality. Hence, in this case symmetric projections $P_{\bar{\psi}} \neq 1-P_{\psi}$ cannot be used. In particular this rules out symmetric projections for one Weyl fermion: $\chi = 1$.

Evidently, the realisation of an even number of only left handed fermions with symmetric projections $P_{\bar{\psi}} = 1-P_{\psi}$ is not excluded by $\chi = n[P_{\psi}] - n[P_{\bar{\psi}}]$. However, if such a theory could be realised, the projections cannot be constant. For constant projections both winding numbers $n[P_{\psi}]$ and $n[P_{\bar{\psi}}]$ vanish and the total chirality is zero, in accordance with the no-go theorem of [6]. Consequently, this additional option, if at all, can only be realised for momentum-dependent chiral projections. In Appendix A we present an example for such a case. The example indicates that, as in the case of doubling modes, $\chi = 2^{2l}$ is needed if one requires invariance under the discrete Euclidean group.

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# A Example for $\chi \neq 0$

In this appendix, we provide an example of a lattice theory with an even number of fermions of the *same* chirality and $P_{\bar{\psi}} = 1-P_{\psi}$. According to Ref. [7], this is not possible on 3-dimensional, spatial lattices. It is not ruled out by (6) for even dimensional space-time lattices, however, and the example shows that it can indeed occur. To keep expressions simple, we actually present a 2-dimensional example and indicate how it can be generalised to 4 dimensions. Our theorem also holds in 2 dimensions, where the winding number (7) takes the form

$$n[P] \equiv \frac{\mathrm{i}}{2\pi} \int_{T^2} \mathrm{tr}\big[P(\mathrm{d}P)^2\big] \tag{46}$$

and the total chirality is given by $\chi = \sum_i \mathrm{tr}\left[P_\psi(k^{(i)})\Gamma_\psi^{(i)}\right]$ with $\Gamma_\psi^{(i)} = \mathrm{i}\Sigma_1^{(i)\dagger}\Sigma_2^{(i)}$. Now consider the naïve Dirac operator

$$D = \gamma_\mu \sin k_\mu \tag{47}$$

(we have set the lattice spacing $a$ to 1) and define chiral projections $P_\psi \equiv \frac{1}{2}(1 - Q_\psi)$ and $P_{\bar{\psi}} \equiv 1 - P_\psi$ with

$$Q_\psi \equiv \frac{\gamma_3 \cos k_1 \cos k_2 + \gamma_1 \sin k_2 - \gamma_2 \sin k_1}{\sqrt{1 + \sin^2 k_1 \sin^2 k_2}} \tag{48}$$

with $\gamma_3 \equiv \mathrm{i}\gamma_1\gamma_2$. It easy to see that $Q_\psi^2 = 1$ and $Q_\psi^\dagger = Q_\psi$, so that $P_{\psi/\bar{\psi}}$ are projections, and that $DQ_\psi = -Q_\psi D$, so that Eq. (5) holds with $P_{\bar{\psi}} = 1 - P_\psi$. Furthermore $Q_\psi$ is analytic in a cylinder around the plane of real $k_i$, so $P_{\psi/\bar{\psi}}$ are as well.

Near the zeros of $D$, $k_\mu^{(i)} = i_\mu \pi$ with the multi-index $i_\mu = 0, 1$,

$$D = (-1)^{i_1}\gamma_1 q_1 + (-1)^{i_2}\gamma_2 q_2 + \mathcal{O}(q^2) \tag{49}$$

where $q_\mu = k_\mu - k_\mu^{(i)}$. We read off $\Sigma_\mu = (-1)^{i_\mu}\gamma_\mu$ and find

$$\Gamma_\psi^{(i)} = (-1)^{i_1+i_2}\gamma_3 \,. \tag{50}$$

On the other hand, Eq. (48) gives

$$Q_\psi(k^{(i)}) = (-1)^{i_1+i_2}\gamma_3 \,. \tag{51}$$

So $P_\psi$ projects onto the left-handed component of $\psi$ at all zeros of $D$. The total chirality after chiral projection is $\chi = -4$, the theory contains 4 species of left-handed fermions.

Equation (48) can be generalised to 4 dimensions as follows: we put $\cos\theta = \cos k_1 \cos k_2 \cos k_3 \cos k_4$, so that $\sin\theta$ still cancels the poles at $\sin k_\mu = 0$. The fraction is the scalar product of a unit vector $t_\mu$ with $\gamma_\mu$. In order for $Q_\psi$ to anticommute with $D$, $t_\mu$ has to be orthogonal to the vector $p_\mu = \sin k_\mu$. This can be achieved with the choice $t = (p_2, -p_1, p_4, -p_3)/|p|$. The resulting projected theory contains 16 species of right-handed fermions. Note that this construction is not possible in 3 dimensions: since $t$ depends only on $p/|p|$ and is orthogonal to $p$, it can be considered as a vector field of unit length on $S^3$; on $S^2$, however, all vector fields vanish somewhere, so such a $t$ can not exist. This argument does not exclude a different construction, of course.

The above example does not contradict Ref. [6]. There, the chiral charge $Q_\psi$ is assumed to be momentum independent.

# B Derivation of (39)

Here we present the technical details of the derivation of (39). In view of (32) and (37), Eq. (39) is equivalent to

$$\lim_{\varepsilon\to 0}\int_{\|k-k^{(i)}\|=\varepsilon} \mathrm{tr}\,\left(P_\psi(k^{(i)})J\right)^{2l-1} = \lim_{\varepsilon\to 0}\int_{\|k-k^{(i)}\|=\varepsilon} \mathrm{tr}\; P_\psi(k^{(i)})J_0^{2l-1}, \tag{52}$$

where $J_0 = |q|^{-2}\, q\cdot\Sigma^\dagger\,\mathrm{d}q\cdot\Sigma$ as defined in (40). We have dropped the index $^{(i)}$ since we focus on one zero of $D$. Only the singular pieces of $J$ can contribute to the integral on the lhs of (52). For tracking them down we write $D$ and $D^{-1}$ in an expansion about $k^{(i)}$ as given in (4a):

$$D(k) = q_\mu\Sigma_\mu + \tilde{M}(k), \qquad D(k)^{-1} = \frac{q_\mu}{|q|^2}\Sigma_\mu^\dagger + M(k), \tag{53}$$

where we have put $q \equiv k-k^{(i)}$ and the matrices $M$ and $\tilde{M}$ have analytic entries. We are only interested in the divergent term in $J = \frac{1}{|q|^2}q\cdot\Sigma^\dagger\,\mathrm{d}D + \mathcal{O}(|q|^0)$. With (4b) it follows that $\Sigma^\dagger = \Pi_\psi\Sigma^\dagger\Pi_{\bar\psi}$. Moreover the projection operators $\Pi_{\psi/\bar\psi}$ commute with $P_\psi(k^{(i)})$ (see (15)). Hence for calculating the lhs of (52) only the singular piece of $J\,\Pi_\psi$ is required. Consequently, we are only interested in the non-vanishing part of $\Pi_{\bar\psi}\,\mathrm{d}D\,\Pi_\psi$. To obtain it, consider

$$q\cdot\Sigma = q\cdot\Sigma D^{-1}D\Pi_\psi = \Pi_{\bar\psi}D\Pi_\psi + q\cdot\Sigma(\tilde{M}\Pi_\psi + q\cdot\Sigma) \tag{54}$$

and

$$q\cdot\Sigma = DD^{-1}q\cdot\Sigma = \tilde{M}\Pi_\psi + \mathcal{O}(q)\;. \tag{55}$$

These imply $\Pi_{\bar\psi}D\Pi_\psi = q\cdot\Sigma + \mathcal{O}(q^2)$, and we find

$$J\,\Pi_\psi = J_0 + \mathcal{O}(|q|^0) \quad \text{with} \quad J_0 \equiv \frac{1}{|q|^2}\,q\cdot\Sigma^\dagger\,\mathrm{d}q\cdot\Sigma\;. \tag{56}$$

Furthermore it follows from (16) that $[J_0,\, P_\psi(k^{(i)})] = 0$. Thus, (52) follows.

# C Chern characters in terms of $P$

We show that the winding number $n[P]$ is given by the integrated Chern character of the fibre bundle associated with $P$, see Eq. (26). To this end, we use

$$F = \mathrm{d}\Psi^\dagger\wedge\mathrm{d}\Psi + \Psi^\dagger\mathrm{d}\Psi\wedge\Psi^\dagger\mathrm{d}\Psi \tag{57}$$

and

$$(\mathrm{d}P)^2\Psi = (\mathrm{d}\Psi\,\Psi^\dagger + \Psi\,\mathrm{d}\Psi^\dagger)^2\Psi = \Psi\,\mathrm{d}\Psi^\dagger\wedge(1-\Psi\Psi^\dagger)\,\mathrm{d}\Psi = \Psi F \tag{58}$$

to find

$$\mathrm{tr}\big[F^l\big] = \mathrm{tr}\big[\Psi^\dagger \Psi F^l\big] = \mathrm{tr}\big[\Psi^\dagger (\mathrm{d}P)^{2l} \Psi\big] = \mathrm{tr}\big[P(\mathrm{d}P)^{2l}\big] \,. \tag{59}$$

The Chern characters can now be expressed as

$$\mathrm{ch}_l(F) \equiv \frac{1}{l!}\,\mathrm{tr}\left[\left(\frac{\mathrm{i}F}{2\pi}\right)^l\right] = \frac{1}{l!}\left(\frac{i}{2\pi}\right)^l \mathrm{tr}\big[P(\mathrm{d}P)^{2l}\big] \,, \tag{60}$$

which coincides with the integrand in the definition (7) of $n[P]$.

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